FEATURES

(Not Very) Complex Math

In Analog Electronics

By
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In this article, I'll use a few specific examples to "uncomplicate" the complex math used in analog electronics, and conduct some experiments to bring more intuition into the analysis.

Analog electronics has a special charm, in my opinion, because it relies on physical properties of matter to achieve the desired performance of the circuits, rather than on digitizing or synthesizing signals in a processor. I have been fascinated by electricity and electronics since I was about 6. Back in the USSR, my father (a physicist) bought me a kit to assemble a radio receiver. This was my first kit that involved soldering and discrete components. To my dismay, there was a stumbling block: I had to make a radio frequency (RF) transformer and an RF choke. This meant winding hundreds of turns of a 0.1mm wire onto a small ferrite toroidal core with an inner diameter of 3mm or so using some DIY winding tool. Well, it was quite impossible for me, hence that receiver was never built…

Despite the toroidal core-induced psychological trauma, my interest in experimenting with analog electronics continued to grow. At the University, we were taught that representing voltage and current as complex numbers greatly simplifies AC circuit analysis. Numerous books repeat the same prescription (for example, *The Art of Electronics* by P. Horowitz & W. Hill, or *Electricity and Magnetism* by E. Purcell); however, these sources do not seem to explain why we can do this. In this article, we will not repeat the college classes on electrical engineering. Instead, we use specific examples to uncomplicate the complex math, and conduct a few experiments to bring more intuition into the analysis. In fact, being able to write and solve a couple of simple equations about the circuit under consideration must be (in my opinion) the first step before running the simulation—for example, SPICE. Otherwise, good luck varying all your inductors, capacitors, and resistors, trying to find the optimal performance in the infinite parameter space!

HELPFUL "IMAGINING"

First, we all know that a parallel LC circuit can be used as a filter exhibiting a high impedance (that is, generalized resistance) in an AC circuit. The frequency of resonance is given by $\omega = 1/\sqrt{(LC)}$. Well, one might say, let us wind just a couple of turns, especially if we are dealing with a small toroidal core and pick appropriate capacitance to achieve the desired resonant frequency. But as always, nothing is simple. The easy approach will not work. Let us see why.

A real inductor in an LC circuit always has a certain resistance due to limited conductivity of the wire (**Figure 1**). Let us power the circuit with a current source, because in practice it will likely be in the collector of a transistor. The physics of the circuit is given with the following equations (some of which are differential):

$$\begin{cases} i_c + i_R = i \\ \frac{q}{c} = u \\ i_R R + \frac{di_R}{dt} L = u \end{cases}$$

where *i* and *u* denote AC current and voltage respectively. The first equation is Kirchhoff's current law for the top node between the resistor and capacitor. The second and the third equations are Kirchhoff's voltage laws for the left and right branches respectively. From a physics perspective, these equations correspond to moving a test charge around a loop of a circuit. The charge gains energy as it moves through sources of electromotive force (EMF) and loses all its gained energy over resistors, capacitors, and inductors. The second equation must be made useful by converting the charge to the current by differentiation. After rearrangement, the final system of equations reads as follows:

$$\begin{cases} i_c + i_R = i \\ \dfrac{i_c}{c} - \dfrac{du}{dt} = 0 \\ i_R R + \dfrac{di_R}{dt} L - u = 0 \end{cases}$$

This system of equations represents the physics of the parallel LCR circuit. In this system, only *i* is known, as it comes from the current source; we purposefully placed the known variable on the right side. In principle, the current may be any function of time—step, sawtooth, square, arbitrary, and so on. Solving such a system (usually numerically) describes the system entirely: all currents and voltages turn into functions of time. In SPICE such a procedure is called "transient analysis."

For radio and similar applications, the usual case is a cosine wave:

$$i = i_a \cos(\omega t)$$

where i_a is the amplitude, ω is the angular frequency, and *t* is time. As one might expect, all other circuits' voltages and currents also behave as cosines with some phase shifts and amplitudes, which need to be found. Solving the system of equations in terms of cosines, amplitudes, and phase shifts will certainly work. However, it may be tedious for bigger and more complex systems. The use of complex math simplifies complex (pun intended) systems of equations by using the following interesting identity, known as Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

Adding a bit to the confusion, in electrical engineering the imaginary unit is written as $j = \sqrt{(-1)}$. Using Mathematica, we can do wonders in symbolic computation. (Mathematica offers various subscription options, including free online access with some limitations.) For those who are interested in inspecting all of the Mathematica notebooks, please refer to the two files on *Circuit Cellar*'s Article Materials and Resources webpage. Let us do a series expansion of the above equation:

$$e^{j\omega t} = 1 + jt\omega - \frac{t^2\omega^2}{2} - \frac{jt^3\omega^3}{6} + \frac{t^4\omega^4}{24} + \frac{jt^5\omega^5}{120} + 0(t^6)$$

Same answer we get for expanding the right side of Euler's formula, as one can prove it directly. Upon assigning $i = i_a e^{j\omega t}$; $i_c = i_{ca} e^{j\omega t}$... with the sub *a* indicating amplitude, and inserting them into the original system, things become easier to work with. A great help comes from the fact that $e^{j\omega t}$ may be removed altogether, because it appears in front of every *i* and *u*:

$$\begin{cases} i_{ca} + i_{Ra} = i_a \\ \dfrac{i_c}{c} - j\omega = 0 \\ i_{Ra}(R + j\omega L) - u_a = 0 \end{cases}$$

Let us make it even more beautiful by representing this algebraic system in a matrix form, having the order of unknowns as i_{ca}, i_{Ra}, u_a. Zeroes correspond to the variable absent in the equation, for example, u_a in the first equation:

$$\begin{pmatrix} 1 & 1 & 0 \\ c^{-1} & 0 & -j\omega \\ 0 & R + j\omega L & -1 \end{pmatrix} \begin{pmatrix} i_{ca} \\ i_{Ra} \\ U_a \end{pmatrix} = \begin{pmatrix} i_a \\ 0 \\ 0 \end{pmatrix}$$

The linear algebraic system has only one solution for i_{ca}, i_{Ra}, u_a. Each amplitude comes out as an expression with inductance, capacitance, resistance, and an imaginary unit. For example:

$$u_a = \frac{i_a(L\omega - jR)}{c\omega(R + jL\omega) - j}$$

BACK TO THE REALS

Now we arrived at "the moment of truth." Let us verify if our solution matches with physics:

1. Multiply each amplitude by $\cos(\omega t) + j\sin(\omega t)$, and expand.
2. Take real parts of each expression, for example:

$$u = \frac{i_a\left(\omega\sin(t\omega)(cL^2\omega^2 + cR^2 - L) + R\cos(t\omega)\right)}{c^2\omega^2(L^2\omega^2 + R^2) - 2cL\omega^2 + 1}$$

3. Insert the real parts of each expression back into the original system of equations representing the physics of the circuit.

Amazingly, the system of equations turns out to be true. Refer to the Mathematica notebook for every equation validated. Thus, in our analytical journey, we ventured out into the domain of complex numbers, conducted rather simple algebraic manipulations to find the only possible solution, and

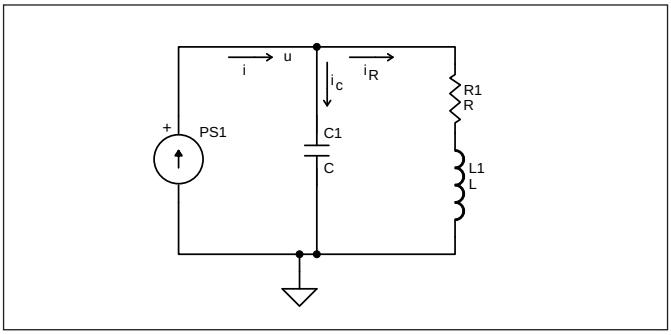
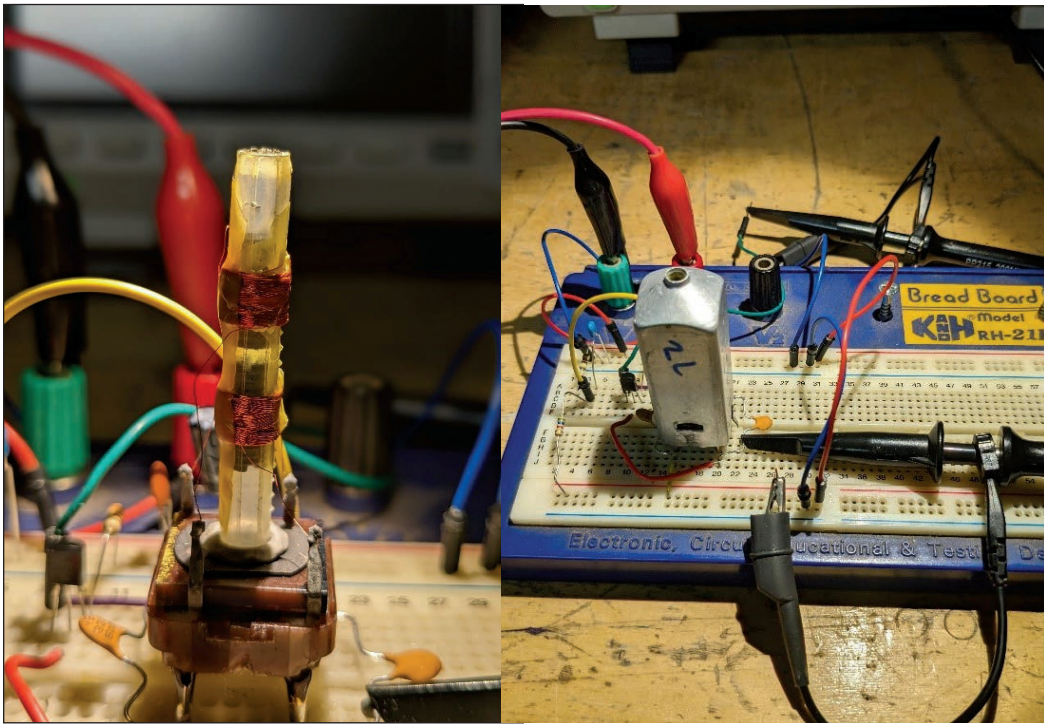


FIGURE 1
LCR circuit powered with a current source.

FIGURE 2
Variable coupling FR transformer in a prototype board. Notice two cores in each coil; the cores can be brought closer to one another.



returned to the real domain. The real part of the solution is also a solution of the original system of differential equations.

Let us go back into the complex domain and inspect the impedance of the circuit, $Z = u_a / i_a$ (see above the complex representation of the voltage amplitude). Again, after doing symbolic rearrangements, let us represent impedance with the real and imaginary parts separated, and let's remove the imaginary unit from the denominator:

$$Z = \frac{R + j\left(L\omega - cR^2\omega - cL^2\omega^3\right)}{c^2R^2\omega^2 + \left(cL\omega^2 - 1\right)^2}$$

At resonance (for which there are several definitions [1]), the phase shift between voltage and current is zero, which means that Z will have only a real part. This is because still having j in the expressions brings $\sin(\omega t)$ from Euler's formula; the sum of sines and cosines does produce a phase shift. There are two physically meaningful frequencies ω at which the imaginary part of Z vanishes: 0 and

$$\frac{\sqrt{L - cR^2}}{\sqrt{cL}}$$

This makes perfect sense for the first solution: at DC, there is no phase shift, because the circuit acts as a resistor. At the other frequency, the impedance is

$$Z_{res} = \frac{L}{cR}$$

This "sad" result tells us that with R on a scale of 1 to 10Ω in a practical radio frequency inductor, the value of L must be about 1000 times larger than the value of c to have a decent impedance at the resonant frequency. Well, this means that a couple of turns on a small toroid and a big capacitor won't do us any good.

TWO COUPLED LCS

Now let us do some practical experiments. A trip to a local surplus store resulted in purchasing an array of unknown LC filters and RF transformers. They date back to the Cold War era and the vacuum tube equipment emanates the "smell of ether." These devices

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Additional materials from the author are available at: www.circuitcellar.com/article-materials

References [1] to [2] as marked in the article can be found there.

are quite old, so their characteristics are nowhere to be found. One particular RF transformer with variable coupling has caught my attention (**Figure 2**).

The DC resistance of each coil was 11.4Ω and 11.8Ω; we can consider them the same at an average value of 11.6Ω. To determine the inductances of each coil, the cores were driven as far apart as possible, yet without leaving any unfilled space within each coil. Using a signal generator (Rigol) set at the output impedance of 100Ω and the voltage amplitude 1.000V_{pp}, the frequency was varied until the voltage across each coil reached 1.00V_{pp} measured by an oscilloscope (Siglent). The obtained frequencies were 16.3 and 16.8kHz, which are quite similar—16.5kHz average. Based on the known formula for the induction impedance, $X_L = jL\omega$ and $\omega = 2\pi f$, the inductance of each coil may be considered as 965μH.

What will happen with the impedance of the circuit, if each coil is involved in a parallel LC circuit with capacitors of the same value, effectively being tuned to the same resonant frequency? How does coupling strength between the coils affect the impedance?

The overall schematic of the experimental setup is shown in **Figure 3**. The coupled LC circuits are in the collector path of a transistor amplifier, hence the current source. The system of equations in the matrix form is as follows:

$$\begin{pmatrix} -1 & 0 & L_1 j\omega + R_1 & -k\sqrt{L_1 L_2 j\omega} & 0 \\ c_1 j\omega & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -k\sqrt{L_1 L_2 j\omega} & L_2 j\omega + R_2 & 1 \\ 0 & 0 & 0 & -1 & c_2 j\omega \end{pmatrix} \begin{pmatrix} u_{1a} \\ i_{c1a} \\ i_{L1a} \\ i_{L2a} \\ u_{2a} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ i_a \\ 0 \\ 0 \end{pmatrix}$$

where k is a coupling coefficient relating the amount of magnetic flux from one coil into another [2], and $R_{1,2}$ are the resistances of the inductors. The first row of the matrix corresponds to Kirchhof's voltage law for one of the LC circuits, whereby the voltage u_1 equals the sum of voltages on the resistor R_1 and L_1 minus the EMF induced by the second inductor. The other rows similarly follow Kirchhof's laws.

As we set out to investigate the impedance of the magnetically coupled circuit having the same inductances and capacitances, let us assign $L_{1,2}$ as L, $c_{1,2}$ as c, $R_{1,2}$ as 10 (the value of the order of magnitude of the resistance in practical coils, like ours).

Following the same solution process as described above (refer to the Mathematica notebook if interested), we obtain

$$Z = \frac{u_{1a}}{i_a}$$

which is a rather big expression when we substitute u_{1a} with its solution from the system of equations. At the resonance, the imaginary part of the impedance must vanish, which translates into setting the numerator of the impedance's imaginary part to zero:

$$\omega\left[-100c + L + \left(-10000c^3 + 300c^2 L + c\left[-3 + k^2\right]L^2\right)\omega^2 + \left(-200c^3\left[1 + k^2\right]L^2 - 3c^2\left[-1 + k^2\right]L^3\right)\omega^4 - c^3\left(-1 + k^2\right)^2 L^4 \omega^6\right] = 0$$

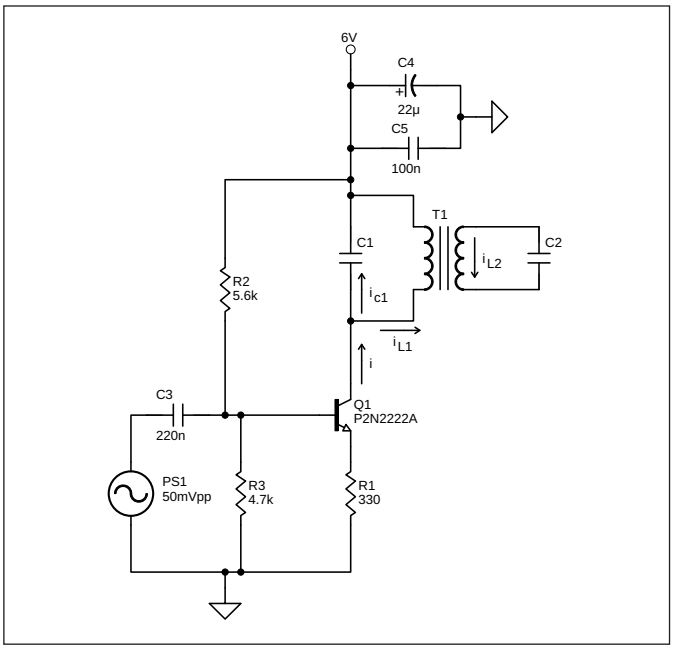

FIGURE 3
Coupled LC circuits testing setup. C1 = C2 (when present) = 330pF; inductances of the coils is 965μH.

It turns out that despite each LC circuit being tuned to the same frequency, multiple (as much as seven) resonances are possible. With $\omega = 0$, the case is trivial; the other resonances occur as solutions of a cubic equation with respect to ω^2. Certainly, some of the solutions will be negative or not real, but nevertheless, more than one additional resonance is possible. **Table 1** shows actual measurements at detected resonances. Note that the resonances were ascertained as a phase shift of 180 degrees between the T1 base and collector voltages, which means zero phase shift between the collector current and the voltage on the lower plate of C1 with respect to +6V.

Clearly, Case 3 reveals two resonances. How do these measurements compare to calculations? Case 2 provides us with k = 0.44/1.80. Placing k and other actual values of R = 11.6Ω, L = 965μH, and c = 330pF into the Mathematica notebook results in predicted physically meaningful resonant frequencies of 252.8, 290.9, and 324.4kHz. Well, we detected the first two in our measurements.

Editors Note: Other articles by Dr. Alexander Pozitkov, PhD., that you may enjoy:

- "Level Switch for Non-Conductive Liquids," *CC* 390, Jan '23.
- "PLC-Based Swimming Pool System: The PoolOS Project ," *CC* 378, Jan '22.
- "Taming Your Wind Turbine: Power Perfected," *CC* 347, June '19.
- "Sensor-Based Smart Air Convector," *CC* 310, May 2016.
- "The NakedCPU (Pt. 2): Experimentation and Communication," *CC* 260, Mar '12.
- "The NakedCPU (Pt. 1): Hardware Experiments and a Roadmap for Navigating Hardware Documentation," *CC* 259, Feb '12.